

COMPARISON OF MODELS FOR (∞, n) -CATEGORIES, II

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ABSTRACT. In this paper we complete a chain of explicit Quillen equivalences between the model category for Θ_{n+1} -spaces and the model category of small categories enriched in Θ_n -spaces.

1. INTRODUCTION

In [17], the second-named author developed the theory of Θ_n -spaces as models for (∞, n) -categories and showed that they are the fibrant objects in a cartesian model category. One of the expected properties of a good model for (∞, n) -categories is that categories enriched in them should be models for $(\infty, n+1)$ -categories. A natural conjecture was that this property should indeed hold for Θ_n -spaces.

The first step in proving this conjecture was establishing a model structure on the category of small categories enriched in Θ_n -spaces. Because the model category for Θ_n -spaces is sufficiently nice (with the cartesian property being particularly key), we were able to use a theorem of Lurie to obtain the desired model structure, as we showed in [9].

While a direct Quillen equivalence between the two model categories was not expected, there was a conjectured chain of Quillen equivalences between intermediate model categories, mostly following by analogy from the case of $(\infty, 1)$ -categories, where the simplicial category and complete Segal space model categories were shown to be equivalent [8]. In [9], we established the first two Quillen equivalences in this chain, proving that categories enriched in Θ_n -spaces are equivalent to Segal category objects in Θ_n -spaces.

In this paper, we complete this conjectured chain of Quillen equivalences. We prove that there is a Quillen equivalence of model categories between Segal category objects and complete Segal objects in Θ_n -spaces. We also establish one final Quillen equivalence between complete Segal objects in Θ_n -spaces and Θ_{n+1} -spaces which was not present in the case of $(\infty, 1)$ -categories.

The final Quillen equivalence is in fact the first in a chain interpolating between the Θ_n -space model and the n -simplicial model for (∞, n) -categories as defined by Barwick and Lurie [14], [15]. Both models were already known to be equivalent on the level of $(\infty, 1)$ -categories using the axiomatic methods of Barwick and Schommer-Pries [5], but we give an alternative approach using an explicit Quillen

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equivalence on the level of model categories. Another approach on the level of $(\infty, 1)$ -categories is taken by Haugseng [11].

Much of the present paper is worked in a substantially more general context. The categories Θ_n are a special case of categories formed by the Θ -construction, and the model category framework of [17] is done in the more general setting. Here, we define complete Segal objects in this same degree of generality and make the comparison accordingly. However, more general Segal category objects present considerable technical difficulties, so our comparison there is only between Segal category objects and complete Segal objects in Θ_n -spaces.

Many other approaches to (∞, n) -categories are being investigated; some references include [1], [2], [3], [4], [13], [14], [15], [16], [19], [20]. We give a more substantial survey in the first paper in this series [9].

We give a brief outline of the contents of the paper. In Section 2, we set up some notation and review basic facts about simplicial presheaf categories and cartesian structures. Section 3 is concerned with the Θ -construction and relations between various presheaf categories. In Sections 4 and 5, we give the model structures for Θ -objects and complete Segal objects, respectively. In Section 6, we give a Quillen equivalence between these model structures; Section 7 states the results in the special case of models for (∞, n) -categories. We consider various notions of equivalence of Segal space objects in Section 8, and these results are used in Section 9 to give the Quillen equivalence between Segal category objects and complete Segal objects.

2. NOTATION AND BACKGROUND

2.1. Simplicial objects and presheaf categories. We write Δ for the simplicial indexing category, with objects

$$[n] = \{0 \leq 1 \leq \cdots \leq n\}$$

and morphisms weakly order-preserving maps. We write δ^{k_0, \dots, k_m} for the map $[m] \rightarrow [n]$ sending $i \mapsto k_i$.

We sometimes extend the notation for objects in the following way. For $p \leq q$ we write $[p, q] = \{p \leq p+1 \leq \cdots \leq q\}$. As an object of Δ , it is identified with $[q-p]$; the notation is meant to indicate that it is equipped with a distinguished inclusion $[p, q] \rightarrow [m]$ for $m \geq q$, sending i to i .

Let $sSets$ denote the category of simplicial sets, or functors $\Delta^{op} \rightarrow Sets$. For a small category \mathcal{C} , let $sPSh(\mathcal{C})$ denote the category of presheaves of simplicial sets on \mathcal{C} , or functors $\mathcal{C}^{op} \rightarrow sSets$. We write $F = F_{\mathcal{C}}: \mathcal{C} \rightarrow sPSh(\mathcal{C})$ for the Yoneda embedding. It is defined by the representable object $F_{\mathcal{C}}(c) = \text{Hom}(-, c)$ for any object c of \mathcal{C} , where the right-hand side is regarded as a discrete simplicial set.

Given a functor $f: \mathcal{C} \rightarrow \mathcal{D}$, we have a sequence of adjoint functors $f_{\#} \dashv f^* \dashv f_*$ of the form

$$\begin{array}{c} sPSh(\mathcal{C}) \\ \downarrow f_{\#} \quad \uparrow f^* \quad \downarrow f_* \\ sPSh(\mathcal{D}) \end{array}$$

where f^* is the restriction functor, defined by $(f^*X)(c) = X(fc)$ for any object c of \mathcal{C} . We note that there is a natural isomorphism

$$f_{\#}Fc \xrightarrow{\cong} F_D: \mathcal{C} \rightarrow s\mathcal{PSh}(\mathcal{D})$$

obtained using the adjointness of $f_{\#}$ and f^* .

We make use of the following observation: if $f: \mathcal{C} \rightleftarrows \mathcal{D} : g$ is an adjoint pair, then there are natural isomorphisms $f^* \cong g_{\#}$ and $f_* \cong g^*$, and thus an adjoint sequence of the form $f_{\#} \dashv f^* \cong g_{\#} \dashv f_* \cong g^* \dashv g_*$.

Let 1 denote the category with a single object and no non-identity morphisms, suppose that \mathcal{C} is equipped with a terminal object $t_{\mathcal{C}}$, and consider the resulting adjoint pair $p: \mathcal{C} \rightleftarrows 1 : t$, where $t(1) = t_{\mathcal{C}}$. Because $p \circ t = \text{id}$, the functor $p^* \cong t_{\#}: s\mathcal{Sets} = s\mathcal{PSh}(1) \rightarrow s\mathcal{PSh}(\mathcal{C})$, which sends a simplicial set X to the constant functor with value X , is fully faithful. Thus $s\mathcal{Sets}$ is equivalent to the full subcategory of constant functors in $s\mathcal{PSh}(\mathcal{C})$, and we sometimes make this identification silently.

More generally, if \mathcal{C} and \mathcal{D} each have terminal objects, then we obtain full embeddings $\pi_{\mathcal{C}}^*: s\mathcal{PSh}(\mathcal{C}) \rightarrow s\mathcal{PSh}(\mathcal{C} \times \mathcal{D})$ and $\pi_{\mathcal{D}}^*: s\mathcal{PSh}(\mathcal{D}) \rightarrow s\mathcal{PSh}(\mathcal{C} \times \mathcal{D})$, induced by the two projection functors.

We are most interested in the categories Δ and Θ_n , both of have terminal objects, so $s\mathcal{PSh}(\Delta)$ and $s\mathcal{PSh}(\Theta_n)$ can be regarded as full subcategories of $s\mathcal{PSh}(\Delta \times \Theta_n)$. In particular, we regard both Yoneda functors F_{Δ} and F_{Θ_n} as objects of $s\mathcal{PSh}(\Delta \times \Theta_n)$, and $F_{\Delta \times \Theta_n}([m], \theta) \cong F_{\Delta}([m]) \times F_{\Theta_n}$. For simplicity, we write F_{Θ} for F_{Θ_n} .

Unless indicated otherwise, when considering $s\mathcal{PSh}(\mathcal{C})$, we assume it has the injective model structure. We often localize this model structure with respect to a set of maps \mathcal{S} , and denote the resulting model structure by $s\mathcal{PSh}(\mathcal{C})_{\mathcal{S}}$. An \mathcal{S} -fibrant object is an injective fibrant object X such that $\text{Map}_{s\mathcal{PSh}(\mathcal{C})}(f, X)$ is a weak equivalence for all $f \in \mathcal{S}$. An \mathcal{S} -local object is any object levelwise weakly equivalent to an \mathcal{S} -fibrant object. We write $\overline{\mathcal{S}}$ for the class of weak equivalences in this model category. The pair $(\mathcal{C}, \mathcal{S})$ of a small category \mathcal{C} together with a set \mathcal{S} of maps in $s\mathcal{PSh}(\mathcal{C})$ is sometimes called a *presentation*.

Throughout this paper morphisms between two objects can take several different forms. By Hom we denote morphism set, by Map or map we denote morphism simplicial set, and by $\underline{\text{Map}}$ or $\underline{\text{map}}$ we denote internal hom object in the relevant category. (In some places we have additional notation to clarify the category in which we take this internal hom; this notation will be explained as needed.)

2.2. Cartesian structures. The additional structure of a cartesian model category is a key feature of some of the model categories which we consider in this paper. We give a brief review of the definitions here, and the reader is referred to [17, §2] for a detailed treatment.

Definition 2.3. A category \mathcal{C} is *cartesian closed* if it has finite products and, for any two objects X and Y of \mathcal{C} , an internal function object Y^X , together with a natural isomorphism

$$\text{Hom}_{\mathcal{C}}(Z, Y^X) \cong \text{Hom}_{\mathcal{C}}(Z \times X, Y)$$

for any third object Z of \mathcal{C} .

If a cartesian closed category additionally has a model structure, we can ask if these two structures are compatible, in the following sense.

Definition 2.4. A model category \mathcal{M} is *cartesian* if its underlying category is cartesian closed, the terminal object is cofibrant, and the following equivalent conditions hold.

- (1) If $f: A \rightarrow A'$ and $g: B \rightarrow B'$ are cofibrations in \mathcal{M} , then the induced map

$$h: A \times B' \coprod_{A \times B} A' \times B \rightarrow A' \times B'$$

is a cofibration. If either f or g is a weak equivalence then so is h .

- (2) If $f: A \rightarrow A'$ is a cofibration and $p: X \rightarrow X'$ is a fibration in \mathcal{M} , then the induced map

$$q: (X')^{A'} \rightarrow (X')^A \times_{X^A} X^{A'}$$

is a fibration. If either f or p is a weak equivalence, then so is q .

We use the following proposition, which was proved for simplicial spaces in [18, 9.2].

Proposition 2.5. [17, 2.10, 2.22] *Let \mathcal{C} be a small category and $F_{\mathcal{C}}: \mathcal{C} \rightarrow s\mathcal{PSh}(\mathcal{C})$ the Yoneda embedding. Consider the category $s\mathcal{PSh}(\mathcal{C})$ with the injective model structure and consider its localization with respect to a set \mathcal{S} of maps. Suppose that if W is an \mathcal{S} -fibrant object, then so is $W^{F_{\mathcal{C}}(c)}$ for any object c of \mathcal{C} . Then the localized model structure is compatible with the cartesian closure.*

Definition 2.6. A presentation $(\mathcal{C}, \mathcal{S})$ is a *cartesian presentation* if, for every \mathcal{S} -local object X in $s\mathcal{PSh}(\mathcal{C})$ and $s: S \rightarrow S'$ in \mathcal{S} , the induced map $Y^s: Y^{S'} \rightarrow Y^S$ is a levelwise weak equivalence, where $X \rightarrow Y$ is a fibrant replacement.

Proposition 2.7. [17, 2.11] *Let $(\mathcal{C}, \mathcal{S})$ be a presentation. Then the following are equivalent.*

- (1) $(\mathcal{C}, \mathcal{S})$ is a cartesian presentation.
- (2) For any \mathcal{S} -fibrant X in $s\mathcal{PSh}(\mathcal{C})$ and any $\text{cin ob}(\mathcal{C})$, the object X^{F_c} is \mathcal{S} -local.
- (3) For any $s: S \rightarrow S'$ in \mathcal{S} and any $c \in \text{ob}(\mathcal{C})$, the map $s \times F_c: S \times F_c \rightarrow S' \times F_c$ is in $\overline{\mathcal{S}}$.

3. THE Θ -CONSTRUCTION AND COMPARISON FUNCTORS

In this section we recall the Θ -construction to obtain a new category $\Theta\mathcal{C}$ from a category \mathcal{C} and set up a comparison between $\Theta\mathcal{C}$ and $\Delta \times \mathcal{C}$. Some of the material in this section can be found in [17].

We begin by recalling the Θ -construction on a small category. Let \mathcal{C} be a small category, and define the category $\Theta\mathcal{C}$ to have objects $[m](c_1, \dots, c_m)$ where $[m]$ is an object of Δ and each c_i is an object of \mathcal{C} . A morphism

$$[m](c_1, \dots, c_m) \rightarrow [p](d_1, \dots, d_p)$$

is given by a morphism $\delta: [m] \rightarrow [p]$ in Δ together with a morphism $f_{ij}: c_i \rightarrow d_j$ in \mathcal{C} for every $1 \leq i \leq m$ and $1 \leq j \leq p$ satisfying $\delta(i-1) < j \leq \delta(i)$. Notice that $\Theta\mathcal{C}$ has a terminal object $[0]$.

Example 3.1. Let Θ_0 be the trivial category with one object and the identity morphism only. Inductively define $\Theta_n = \Theta\Theta_{n-1}$. The categories Θ_n were first defined by Joyal; the inductive definition described here is due to Berger [7] and also given in [17]. Observe that $\Theta_1 = \Delta$.

Berger proved also that each Θ_n is a Reedy category, so the category $s\mathcal{P}Sh(\Theta_n)$ can be equipped with the Reedy model structure, in which the weak equivalences are given by levelwise weak equivalences of simplicial structure [6]. In [10], we prove that Θ_n has the additional structure of an elegant Reedy category, so that the Reedy model structure is exactly the injective model structure.

3.2. Functors involving $\Delta \times \mathcal{C}$ and $\Theta\mathcal{C}$. Let \mathcal{C} be a small category with terminal object $t = t_{\mathcal{C}}$. We consider the categories $\Delta \times \mathcal{C}$ and $\Theta\mathcal{C}$. Both these categories have canonical functors to Δ ; in fact, we have adjoint pairs

$$\pi_{\Delta} : \Delta \times \mathcal{C} \rightleftarrows \Delta : \tau_{\Delta} \quad \text{and} \quad \pi_{\Theta} : \Theta\mathcal{C} \rightleftarrows \Theta 1 = \Delta : \tau_{\Theta}$$

defined by $\pi_{\Delta} = \text{id}_{\Delta} \times p$, $\tau_{\Delta} = \text{id}_{\Delta} \times t$, $\pi_{\Theta} = \Theta p$, and $\tau_{\Theta} = \Theta t$. Explicitly on objects, we have

$$\begin{aligned} \pi_{\Delta}([m], c) &= [m] & \pi_{\Theta}([m](c_1, \dots, c_m)) &= [m] \\ \tau_{\Delta}([m]) &= ([m], t) & \tau_{\Theta}([m]) &= [m](t, \dots, t). \end{aligned}$$

On the level of simplicial presheaves, these functors allow us to have an *underlying simplicial space* associated to an object of $s\mathcal{P}Sh(\Theta\mathcal{C})$ or $s\mathcal{P}Sh(\Delta \times \mathcal{C})$; further evaluating this underlying simplicial space at degree zero gives an *underlying space*.

Let $d : \Delta \times \mathcal{C} \rightarrow \Theta\mathcal{C}$ be the functor defined on objects by

$$d([m], c) := [m](c, \dots, c).$$

Note that $\pi_{\Theta}d = \pi_{\Delta}$ and $d\tau_{\Delta} = \tau_{\Theta}$.

The functor d induces the sequence of adjoint functors on presheaves

$$\begin{array}{c} s\mathcal{P}Sh(\Delta \times \mathcal{C}) \\ \downarrow d_{\#} \quad \uparrow d^* \quad \downarrow d_* \\ s\mathcal{P}Sh(\Theta\mathcal{C}) \end{array}$$

These adjoint pairs are in fact Quillen pairs

$$d_{\#} : s\mathcal{P}Sh(\Delta \times \mathcal{C}) \rightleftarrows s\mathcal{P}Sh(\Theta\mathcal{C}) : d^*$$

and

$$d^* : s\mathcal{P}Sh(\Theta\mathcal{C}) \rightleftarrows s\mathcal{P}Sh(\Delta \times \mathcal{C}) : d_{\#}$$

on the projective and injective model structures, respectively. While our primary interest is in the second Quillen pair, the first will be useful for making calculations.

Proposition 3.3. *There are natural isomorphisms $\pi_{\Theta}^* d^* \cong \pi_{\Delta}^*$ of functors $s\mathcal{P}Sh(\Delta \times \mathcal{C}) \rightarrow s\mathcal{P}Sh(\Delta)$, and natural isomorphisms $\pi_{\Delta}^* d_{\#} \cong \pi_{\Theta}^*$ of functors $s\mathcal{P}Sh(\Theta\mathcal{C}) \rightarrow s\mathcal{P}Sh(\Delta)$.*

Proof. The first isomorphism follows immediately from the fact that $\pi_{\Theta}d = \pi_{\Delta}$. The second follows from the fact that $d\tau_{\Delta} = \tau_{\Theta}$ and the observation that $\pi_{\Delta}^* \cong (\tau_{\Delta})_{\#}$ and $\pi_{\Theta}^* \cong (\tau_{\Theta})_{\#}$ from Section 2.1. \square

Explicitly, for any object W of $s\mathcal{P}Sh(\Delta \times \mathcal{C})$, we have

$$(d_{\#}W)([m](t, \dots, t)) \cong W([m], t),$$

and for any object X of $s\mathcal{P}Sh(\Theta\mathcal{C})$, we have

$$(d^*X)([m], t) \cong X([m](t, \dots, t)).$$

That is, the functors $d_{\#}$ and d^* both preserve underlying simplicial space, up to isomorphism.

In particular, $(d_{\#}W)([0]) \cong W([0], t)$ and $(d^*X)([0], t) \cong X([0])$. Since right adjoints preserve terminal objects, we know that $d^*(F_{\Theta\mathcal{C}}[0]) \cong F_{\Delta}[0] \times F_{\mathcal{C}}(t)$, from which we see that $(d^*W)([0]) \cong W([0], t)$. That is, the functors $d_{\#}$, d^* , and d_* all preserve underlying space, up to isomorphism.

3.4. The intertwining functor. Recall from [17, 4.4] the *intertwining functor*

$$V: \Theta(s\mathcal{P}Sh(\mathcal{C})) \rightarrow s\mathcal{P}Sh(\Theta\mathcal{C}),$$

defined on objects by

$$\begin{aligned} V[m](X_1, \dots, X_m)([k](c_1, \dots, c_k)) &:= \text{Map}_{\Theta(s\mathcal{P}Sh(\mathcal{C}))}([k](Fc_1, \dots, Fc_k), [m](X_1, \dots, X_m)) \\ &\cong \prod_{\delta: [k] \rightarrow [m]} \prod_{i=1}^k \prod_{j=\delta(i-1)+1}^{\delta(i)} X_j(c_i). \end{aligned}$$

The intertwining functor is the left Kan extension of $F_{\Theta\mathcal{C}}$ along $\Theta F_{\mathcal{C}}$.

More generally, if $f: K \rightarrow F[m]$ is a map in $sSets$, regarded as a maps of discrete objects in $s\mathcal{P}Sh(\Theta\mathcal{C})$, we define

$$V_f[m](X_1, \dots, X_m) := \lim \left[V[m](X_1, \dots, X_m) \rightarrow F[m] \xleftarrow{f} K \right],$$

so that

$$V_f[m](X_1, \dots, X_m)([k](c_1, \dots, c_k)) \cong \prod_{\delta: F[k] \rightarrow K} \prod_{i=1}^k \prod_{j=f\delta(i-1)+1}^{f\delta(i)} X_j(c_i).$$

If $K \subseteq F[m]$ is a subobject, we often write $V_K[m](X_1, \dots, X_m)$ for $V_f[m](X_1, \dots, X_m)$.

We recall that

$$\begin{aligned} V_f[m](X_1, \dots, X_{d-1}, \emptyset, X_{d+1}, \dots, X_m) \\ \cong V_{f \times_{F[m]} F[0, d-1]}[m](X_1, \dots, X_{d-1}) \amalg V_{f \times_{F[m]} F[d, m]}[m](X_{d+1}, \dots, X_m), \end{aligned}$$

and that

$$V_f[m](X_1, \dots, X_{d-1}, -, X_{d+1}, \dots, X_m): s\mathcal{P}Sh(\mathcal{C}) \rightarrow V_f[m](X_1, \dots, X_{d-1}, \emptyset, X_{d+1}, \dots, X_m) \backslash s\mathcal{P}Sh(\Theta\mathcal{C})$$

is a left adjoint (and thus is colimit preserving), where the category on the right is the category of objects of $s\mathcal{P}Sh(\Theta\mathcal{C})$ under $V_f[m](X_1, \dots, X_{d-1}, \emptyset, X_{d+1}, \dots, X_m)$.

In particular, the functor

$$V[1]: s\mathcal{P}Sh(\mathcal{C}) \rightarrow (F[0] \amalg F[0]) \backslash s\mathcal{P}Sh(\Theta\mathcal{C})$$

is a left adjoint. Its right adjoint $(\partial F[1] \xrightarrow{(x_0, x_1)} X) \mapsto M_X^{\Theta}(x_0, x_1)$ is described below. A short calculation gives the following useful fact.

Proposition 3.5. *For any object c of \mathcal{C} , there is an isomorphism $V[1](F_{\mathcal{C}}(c)) \cong F_{\Theta}([1](c))$ in $s\mathcal{P}Sh(\Theta\mathcal{C})$.*

We also have that if $f: K \rightarrow F[m]$ is a colimit of a diagram $J \rightarrow sSets/F[m]$ which sends an object j of J to a map $f_j: K_j \rightarrow F[m]$, we have

$$V_f[m](X_1, \dots, X_m) \cong \text{colim}_{j \in J} V_{f_j}[m](X_1, \dots, X_m),$$

the latter colimit taking place in $s\mathcal{P}Sh(\Theta\mathcal{C})$.

3.6. Relation between comparison and intertwining. It is important to understand $d^*(V_K[m](X_1, \dots, X_m))$, where $K \subseteq F[m]$. We have

$$\begin{aligned} d^*(V_K[m](X_1, \dots, X_m))([k], c) &\cong \prod_{\delta: F[k] \rightarrow K} \prod_{i=1}^k \prod_{j=\delta(i-1)+1}^{\delta(i)} X_j(c) \\ &\cong \prod_{\delta: F[k] \rightarrow K} \prod_{j=\delta(0)+1}^{\delta(k)} X_j(c). \end{aligned}$$

In particular,

$$d^*(V[0]) \cong F[0] = 1,$$

and

$$\begin{aligned} d^*(V[1](X)) &\cong \operatorname{colim} \left(\coprod_{\delta: [k] \rightarrow [1]} X(c) \leftarrow X(c) \amalg X(c) \rightarrow F[0] \amalg F[0] \right) \\ &\cong F[1] \times X \cup_{\partial F[1] \times X} \partial F[1]. \end{aligned}$$

We write $\Sigma: s\mathcal{P}Sh(\mathcal{C}) \rightarrow s\mathcal{P}Sh(\Delta \times \mathcal{C})$ for the functor $\Sigma(X) := d^*(V[1](X))$; it comes with a distinguished map $\partial F[1] \rightarrow \Sigma(X)$.

Proposition 3.7. *There is a natural isomorphism $d_{\#}(\Sigma X) \cong V[1](X)$, compatible with the natural isomorphism $d_{\#}(\partial F[1]) \cong V[0] \amalg V[0]$.*

Proof. The functors

$$V[1], d_{\#}\Sigma: s\mathcal{P}Sh(\mathcal{C}) \rightarrow (F[0] \amalg F[0]) \backslash s\mathcal{P}Sh(\Theta\mathcal{C})$$

preserve $s\mathcal{S}ets$ -weighted colimits. The isomorphism $\Sigma \xrightarrow{\cong} d^*V[1]$ is adjoint to a map $\alpha: d_{\#}\Sigma \rightarrow V[1]$. Thus, it suffices to check that α is an isomorphism at representable objects, which holds since

$$d_{\#}\Sigma(Fc) \cong d_{\#}(F[1] \times Fc \cup_{\partial F[1] \times Fc} \partial F[1]) \cong F[1](c) \cup_{F[0] \amalg F[0]} (F[0] \amalg F[0]) \cong F[1](c).$$

□

The following result gives us an inductive description of $d^*(V_f[m](X_1, \dots, X_m))$, as a quotient of $K \times (X_1 \times \dots \times X_m)$.

Proposition 3.8. *Fix a map $f: K \rightarrow F[m]$ of simplicial sets. Let*

$$T_m = F[0, m-1] \cup_{F[1, m-1]} F[1, m],$$

which comes with an inclusion $T_m \rightarrow F[m]$; the object T_m is the union of the first and last faces of the m -simplex along their common intersection. Then for $m \geq 1$ there is a pushout diagram in $s\mathcal{P}Sh(\Delta \times \mathcal{C})$ of the form

$$\begin{array}{ccc} (T_m \times_{F[m]} K) \times (X_1 \times \dots \times X_m) & \longrightarrow & K \times (X_1 \times \dots \times X_m) \\ \downarrow & & \downarrow \\ d^*(V_{T_m \times_{F[m]} K}[m](X_1, \dots, X_m)) & \longrightarrow & d^*(V_K[m](X_1, \dots, X_m)) \end{array}$$

Proof. The proof is by a straightforward calculation. Assume for simplicity that $f = \text{id}_{F[m]}$. Note that maps $\delta: F[k] \rightarrow T_m$ are precisely the maps $\delta: [k] \rightarrow [m]$ such that either $\delta(0) > 0$ or $\delta(k) < m$. If we evaluate the above square at each corner at $([k], c)$, we get

$$\begin{array}{ccc} \coprod_{\delta: F[k] \rightarrow T_m} \prod_{j=1}^m X_j(c) & \longrightarrow & \coprod_{\delta: F[k] \rightarrow F[m]} \prod_{j=1}^m X_j(c) \\ \downarrow & & \downarrow \\ \coprod_{\delta: F[k] \rightarrow T_m} \prod_{j=\delta(0)+1}^{\delta(k)} X_j(c) & \longrightarrow & \coprod_{\delta: F[k] \rightarrow F[m]} \prod_{j=\delta(0)+1}^{\delta(k)} X_j(c) \end{array}$$

which is a pushout. \square

Recall that because $T_m = F[0, m-1] \cup_{F[1, m-1]} F[1, m]$, we have that $V_{T_m} \cong V_{F[0, m-1]} \cup_{V_{F[1, m-1]}} V_{F[1, m]}$, and d^* preserves colimits. Thus the above proposition gives an inductive description of d^*V_f .

We note that since $T_m \rightarrow F[m]$ is a monomorphism, it follows that the above square is actually a homotopy pushout with respect to levelwise weak equivalences in $s\mathcal{PSh}(\Delta \times \mathcal{C})$. Furthermore, since $T_m = F[0, m-1] \cup_{F[1, m-1]} F[1, m]$ presents T_m as a pushout along inclusions, the resulting pushout squares building $(T_m \times_{F[m]} f) \times (X_1 \times \cdots \times X_m)$ and $d^*(V_{T_m \times_{F[m]} f}[m](X_1, \dots, X_m))$ are also homotopy pushouts.

3.9. Mapping objects. Given X in $s\mathcal{PSh}(\Theta\mathcal{C})$ and $x_0, \dots, x_m \in X[0]$, let $M_X^\Theta(x_0, \dots, x_m)$ denote the object of $s\mathcal{PSh}(\mathcal{C}^m)$ defined by

$$M_X^\Theta(x_0, \dots, x_m)(c_1, \dots, c_m) := \lim (X[m](c_1, \dots, c_m) \rightarrow X[0]^{m+1} \leftarrow \{(x_0, \dots, x_m)\}).$$

Equivalently, $M_X^\Theta(x_0, \dots, x_m)(c_1, \dots, c_m)$ is the fiber of

$$\text{Map}_{s\mathcal{PSh}(\Theta\mathcal{C})}(F[m](c_1, \dots, c_m), X) \rightarrow \text{Map}_{s\mathcal{PSh}(\Theta\mathcal{C})}(\coprod_{m+1} F[0], X)$$

over (x_0, \dots, x_m) .

Likewise, given W in $s\mathcal{PSh}(\Delta \times \mathcal{C})$ and $x_0, \dots, x_m \in W([0], t)$, let $M_W^\Delta(x_0, \dots, x_m)$ denote the object of $s\mathcal{PSh}(\mathcal{C})$ defined by

$$M_W^\Delta(x_0, \dots, x_m)(c) := \lim (W([m], c) \rightarrow W([0], c)^{m+1} \leftarrow \{(x_0, \dots, x_m)\})$$

where the second arrow uses $W([0], t) \rightarrow W([0], c)$.

In particular, we note that $M_W^\Delta(x_0, x_1)(c)$ is the fiber of

$$\text{Map}_{s\mathcal{PSh}(\Delta \times \mathcal{C})}(\Sigma(Fc), W) \rightarrow \text{Map}_{s\mathcal{PSh}(\Delta \times \mathcal{C})}(F[0] \amalg F[0], W).$$

The following fact can be verified from the definitions of these mapping objects.

Proposition 3.10. *The functors d^* and d_* are compatible with M^Δ and M^Θ . That is, if W is an object of $s\mathcal{PSh}(\Delta \times \mathcal{C})$ and X is an object of $s\mathcal{PSh}(\Theta \times \mathcal{C})$, then*

$$M_{d_*W}^\Theta(x_0, x_1) \cong M_W^\Delta(x_0, x_1) \quad \text{and} \quad M_{d^*X}^\Delta(y_0, y_1) \cong M_X^\Theta(y_0, y_1)$$

for all $x_0, x_1 \in W([0], t) \cong (d_*W)([0])$ and $y_0, y_1 \in X([0]) \cong (d^*X)([0], t)$.

4. Θ -OBJECTS

Let \mathcal{C} be a small category with terminal object t . In this section, we are interested in a certain localization of $s\mathcal{P}Sh(\Theta\mathcal{C})$ which is described in more detail in [17]. The local objects are those satisfying Segal and completeness conditions, together with local conditions inherited from a localization of $s\mathcal{P}Sh(\mathcal{C})$, given by a set \mathcal{S} of cofibrations in $s\mathcal{P}Sh(\mathcal{C})$.

Recall that for simplicial spaces, the inclusion maps $\varphi^m: G[m] \rightarrow F[m]$ for $m \geq 0$ are used to encode the Segal condition, where

$$G[m] = \operatorname{colim} (F[1] \rightarrow F[0] \leftarrow \cdots \rightarrow F[0] \leftarrow F[1]).$$

Notice that φ^0 and φ^1 are isomorphisms. Additionally, if E denotes the discrete nerve of the groupoid with two objects and a single isomorphism between them, the collapse map $z: E \rightarrow F[0]$ is used to encode the completeness condition.

For the more general case of $s\mathcal{P}Sh(\Theta\mathcal{C})$, the Segal condition is similarly encoded by maps

$$G[m](c_1, \dots, c_m) \rightarrow F[m](c_1, \dots, c_m)$$

for each $m \geq 0$ and m -tuple (c_1, \dots, c_m) of objects of \mathcal{C} , where

$$G[m](c_1, \dots, c_m) = \operatorname{colim} (F[1](c_1) \rightarrow F[0] \leftarrow \cdots \rightarrow F[0] \leftarrow F[1](c_m)).$$

Denote by Se^Θ the set of all such maps. Alternatively, we can use the intertwining functor to define Se^Θ as the set of all maps of the form

$$V_{G[m]}[m](c_1, \dots, c_m) \rightarrow V[m](c_1, \dots, c_m).$$

Definition 4.1. Let $(\mathcal{C}, \mathcal{S})$ be a presentation. An object X of $s\mathcal{P}Sh(\Theta\mathcal{C})$ is a Θ -object relative to \mathcal{S} if:

- (1) X is injective fibrant,
- (2) for all $m \geq 2$ and all tuples c_1, \dots, c_m of objects in \mathcal{C} , the map $(X\delta^{01}, \dots, X\delta^{m-1,m}): X[m](c_1, \dots, c_m) \rightarrow X[1](c_1) \times_{X[0]} \cdots \times_{X[0]} X[1](c_m)$ is a weak equivalence of simplicial sets,
- (3) the object $\tau_\Theta^* X$ of $s\mathcal{P}Sh(\Delta)$ is a complete Segal space, and
- (4) for all $x_0, x_1 \in X[0]$, the object $M_X^\Theta(x_0, x_1)$ in $s\mathcal{P}Sh(\mathcal{C})$ is \mathcal{S} -local.

Remark 4.2. We note that, in the presence of (1) and (4), we can replace condition (2) with the following.

- (2') For all $m \geq 2$, all tuples c_1, \dots, c_m of objects in \mathcal{C} , and all $x_0, \dots, x_m \in X[0]$, the map

$$M_X^\Theta(x_0, \dots, x_m) \rightarrow M_X^\Theta(x_0, x_1) \times \cdots \times M_X^\Theta(x_{m-1}, x_m).$$

is a weak equivalence.

Proposition 4.3. Consider the injective model structure $s\mathcal{P}Sh(\Theta\mathcal{C})$, and the following sets of maps.

- (1) Let $\operatorname{Se}^\Theta = \{V_{G[m]}[m](c_1, \dots, c_m) \rightarrow V[m](c_1, \dots, c_m)\}$, where c_1, \dots, c_m range over all objects of \mathcal{C} and $m \geq 0$.
- (2) Let $\operatorname{Cpt}^\Theta = \{\pi_\Theta^* z\}$, where $z: Z \rightarrow F[0]$ in $s\mathcal{P}Sh(\Delta)$ is the map defining completeness for simplicial spaces.
- (3) Let $\operatorname{Rec}^\Theta(\mathcal{S}) = \{V[1](f)\}$, where f ranges over all elements of \mathcal{S} .

Let $\mathcal{S}^\Theta = \text{Se}^\Theta \cup \text{Cpt}^\Theta \cup \text{Rec}^\Theta(\mathcal{S})$. Then X is a Θ -object if and only if it is injective fibrant and \mathcal{S}^Θ -local.

The names of the sets of maps are meant to suggest, respectively, Segal maps, the completeness map, and recursive maps, which capture localizations from $s\mathcal{P}Sh(\mathcal{C})_{\mathcal{S}}$.

Theorem 4.4. [17, 6.1, 7.21, 8.1] *The presentations $(\Theta\mathcal{C}, \text{Se}^\Theta)$ and $(\Theta\mathcal{C}, \text{Se}^\Theta \cup \text{Cpt}^\Theta)$ are cartesian. Furthermore, if $(\mathcal{C}, \mathcal{S})$ is cartesian, then so is $(\Theta\mathcal{C}, \mathcal{S}^\Theta)$.*

Proposition 4.5. *Let $f: X \rightarrow Y$ be a map between \mathcal{S}^Θ -local objects in $s\mathcal{P}Sh(\Theta\mathcal{C})$. Then f is in \mathcal{S}^Θ if and only if*

$$f([0]): X[0] \rightarrow Y[0]$$

and

$$f[1](c): X[1](c) \rightarrow Y[1](c)$$

for all objects c of \mathcal{C} are weak equivalences of spaces.

Proof. Recall that a map between \mathcal{S}^Θ -fibrant objects is in $\overline{\mathcal{S}^\Theta}$ if and only if it is a levelwise weak equivalence. For \mathcal{S}^Θ -fibrant objects, the value at a general $[m](c_1, \dots, c_m)$ in $\Theta\mathcal{C}$ is a homotopy limit of values at $[1](c_i)$ and $[0]$, verifying the proposition when X and Y are \mathcal{S}^Θ -fibrant. Since any \mathcal{S}^Θ -local object is levelwise weakly equivalent to a \mathcal{S}^Θ -fibrant object, the proposition follows. \square

Example 4.6. When $\mathcal{C} = \Theta_{n-1}$, so that $\Theta\mathcal{C} = \Theta_n$, then the Θ -objects are called Θ_n -spaces. Since the categories Θ_n are meant to encode higher categories, in the sense that Δ encodes categories, Θ_n -spaces are higher-order versions of complete Segal spaces and hence models for (∞, n) -categories.

5. SEGAL OBJECTS AND COMPLETE SEGAL OBJECTS IN $s\mathcal{P}Sh(\Delta \times \mathcal{C})$

Again, let \mathcal{C} be a small category with terminal object $t = t_{\mathcal{C}}$.

Definition 5.1. Let $(\mathcal{C}, \mathcal{S})$ be a presentation. An object W of $s\mathcal{P}Sh(\Delta \times \mathcal{C})$ is a *Segal object relative to \mathcal{S}* if

- (1) the object W is injective fibrant.
- (2) for all $m \geq 2$ and c objects of \mathcal{C} , the map
$$(W\delta^{01}, \dots, W\delta^{m-1,m}): W([m], c) \rightarrow W([1], c) \times_{W([0], c)} \cdots \times_{W([0], c)} W([1], c)$$
is a weak equivalence of spaces, and
- (3) for all $x_0, x_1 \in W([0], t)$, the object $M_W^\Delta(x_0, x_1)$ in $s\mathcal{P}Sh(\mathcal{C})$ is \mathcal{S} -local.

We prove the following theorem in [9, 3.14] in the case where $\mathcal{C} = \Theta_n$ with the Θ -object localization. Here, we consider the more general case.

Theorem 5.2. *There is a localization of the injective model structure on $s\mathcal{P}Sh(\Delta \times \mathcal{C})$ in which the fibrant objects are precisely the Segal objects. Furthermore, this model structure is cartesian if $(\mathcal{C}, \mathcal{S})$ is a cartesian presentation.*

Proof. The existence of the model structure is obtained using a left Bousfield localization of the Reedy model structure [12, 4.1.1]. We localize with respect to the union of the following two sets of morphisms:

- (1) $\text{Se}^\Delta = \{G[m] \times Fc \rightarrow F[m] \times Fc\}$, for all $c \in \text{ob}(\mathcal{C})$ and $m \geq 2$, and
- (2) $\text{Rec}^\Delta(\mathcal{S}) = \{\Sigma(s): \Sigma(S) \rightarrow \Sigma(S')\}$ for all $(s: S \rightarrow S') \in \mathcal{S}$.

An object W which is local with respect to Se^Δ satisfies that $W(-, c)$ is a Segal space for all objects c of \mathcal{C} , and W which is local with respect to Rec^Δ satisfies that $W([m], -)$ is an \mathcal{S} -local object in $s\mathcal{PSh}(\mathcal{C})$ for all $[m] \in \text{ob}(\Delta)$.

To prove that the model structure is cartesian when $(\mathcal{C}, \mathcal{S})$ is, we apply Proposition 2.5 to $\Delta \times \mathcal{C}$ and use the localization just described. Therefore, we must prove that if W is a Segal object, then so is $W^{F_{\Delta \times \mathcal{C}}([m], c)}$ for any object $([m], c)$ of $\Delta \times \mathcal{C}$.

Since W is assumed to be a Segal object, we know that, for any object $[m]$ of Δ , $W([m], -)$ is a \mathcal{S} -local, and, for any object c of \mathcal{C} , $W(-, c)$ is a Segal space. Since the Segal space model structure and $s\mathcal{PSh}(\mathcal{C})_{\mathcal{S}}$ are cartesian, we know that $W(-, c)^{F_{\Delta}[m]}$ is a Segal space and $W([m], -)^{F_{\mathcal{C}}(c)}$ is \mathcal{S} -local. Therefore, it suffices to prove that

$$W^{F_{\Delta \times \mathcal{C}}([m], t)}(-, \theta) \cong W(-, c)^{F_{\Delta}[m]}$$

for every object $[m]$ of Δ and the terminal object t of \mathcal{C} , and

$$W^{F_{\Delta \times \mathcal{C}}([0], c)}([m], -) \cong W([m], -)^{F_{\mathcal{C}}(c)}$$

for every object c of \mathcal{C} . We prove the first of these isomorphisms; the second is proved analogously.

Let $\underline{\text{Map}}_\Delta$, $\underline{\text{Map}}_{\mathcal{C}}$, and $\underline{\text{Map}}_{\Delta \times \mathcal{C}}$ denote the internal hom objects in $s\mathcal{PSh}(\Delta)$, $s\mathcal{PSh}(\mathcal{C})$, and $s\mathcal{PSh}(\Delta \times \mathcal{C})$, respectively. Then we have isomorphisms on the level of k -simplices

$$\begin{aligned} \left[W(-, c)^{F_{\Delta}[m]} \right]_k &\cong \underline{\text{Map}}_\Delta(F_\Delta[k], W(-, c)^{F_{\Delta}[m]})_0 \\ &\cong \underline{\text{Map}}_\Delta(F_\Delta[k] \times F_\Delta[m], W(-, c))_0 \\ &\cong \underline{\text{Map}}_\Delta(F_\Delta[k] \times F_\Delta[m], ([\ell] \mapsto \text{Map}_{\mathcal{C}}(F_{\mathcal{C}}(c), W([\ell], -)))(t))_0 \\ &\cong \underline{\text{Map}}_{\Delta \times \mathcal{C}}(F_\Delta[k] \times F_\Delta[m] \times F_{\mathcal{C}}(c), W)([0], t) \\ &\cong \underline{\text{Map}}_{\Delta \times \mathcal{C}}(F_\Delta[k] \times F_{\mathcal{C}}(c), W^{F_{\Delta}[m]})([0], t) \\ &\cong \underline{\text{Map}}_{\Delta \times \mathcal{C}}(F_\Delta[k] \times F_{\mathcal{C}}(c), W^{F_{\Delta \times \mathcal{C}}([m], t)})([0], t) \\ &\cong \underline{\text{Map}}_\Delta(F_\Delta[k], W^{F_{\Delta \times \mathcal{C}}([m], t)}(-, c))_0 \\ &\cong \left[W^{F_{\Delta \times \mathcal{C}}([m], t)}(-, c) \right]_k \end{aligned}$$

which proves the desired isomorphism. \square

We now add further conditions to obtain complete Segal objects.

Definition 5.3. Let $(\mathcal{C}, \mathcal{S})$ be a presentation. An object W of $s\mathcal{PSh}(\Delta \times \mathcal{C})$ is a *complete Segal object* relative to \mathcal{S} if:

- (1) the object W is injective fibrant,
- (2) for all $m \geq 2$ and $c \in \text{ob}(\mathcal{C})$, the map $(W\delta^{01}, \dots, W\delta^{m-1, m}): W([m], c) \rightarrow W([1], c) \times_{W([0], c)} \cdots \times_{W([0], c)} W([1], c)$ is a weak equivalence of spaces,
- (3) for all $x_0, x_1 \in W([0], t)$, the object $M_W^\Delta(x_0, x_1)$ in $s\mathcal{PSh}(\mathcal{C})$ is \mathcal{S} -local,
- (4) the object $\tau_\Delta^* W$ of $s\mathcal{PSh}(\Delta)$ is a complete Segal space, and
- (5) for all objects $c \in \mathcal{C}$, the map $W([0], t) \rightarrow W([0], c)$ is a weak equivalence.

Remark 5.4. The need for condition (5) requires some explanation, since it is unnecessary in the case of complete Segal spaces. A simplicial object in any category \mathcal{C} , satisfying the Segal condition, models an internal category \mathcal{C} , where both

the morphisms and the objects have the structure of objects of \mathcal{C} . However, our objective in defining complete Segal objects is to have an up-to-homotopy version of categories enriched in \mathcal{C} , where only the morphisms are objects of \mathcal{C} and the objects form a set.

This feature is more distinct in the structure of Segal category objects, where the degree zero object is forced to be a set. Looking for a moment at the case of $(\infty, 1)$ -categories, the transition from Segal categories (with discrete space in degree zero) to complete Segal spaces (where we drop the discreteness assumption but require completeness) can be thought of as a homotopical change: a set is replaced by a space. Here, we want the same transition. In particular, we want the object in degree zero to be a space, not a more general object of $s\mathcal{P}Sh(\mathcal{C})$.

Remark 5.5. Observe that condition (4) corresponds to the desired weak equivalence between the object in degree zero and the object of homotopy equivalences. We do not immediately get what one might expect, which is the following condition:

(4') $W(-, c)$ is a complete Segal space for any object c in \mathcal{C} .

This condition is in fact a more desirable one, in that it is the one that is compatible with the cartesian structure, and we make use of it in what follows.

In the special case where $\mathcal{C} = \Theta_n$, we expect that any W which satisfies conditions (2)-(4) actually satisfies (4'), and this implication is of substantial interest in its own right. We reserve a proof, however, for future work. We do not expect this implication to hold in general, however.

Theorem 5.6. *There is a model structure on $s\mathcal{P}Sh(\Delta \times \mathcal{C})$ in which the fibrant objects are the complete Segal space objects.*

Proof. Consider the following sets of maps in $s\mathcal{P}Sh(\Delta \times \mathcal{C})$.

- Let $\text{Se}^\Delta = \{G[m] \times Fc \rightarrow F[m] \times Fc\}$, where c ranges over objects of \mathcal{C} , and $m \geq 2$.
- Let $\text{Rec}^\Delta(\mathcal{S}) = \{\Sigma(s): \Sigma(S) \rightarrow \Sigma(S')\}$ for all $(s: S \rightarrow S') \in \mathcal{S}$.
- Let $\text{Cpt}^\Delta = \{\pi_\Delta^* z\}$, where $z: E \rightarrow F[0]$ is the map defining completeness in $s\mathcal{P}Sh(\Delta)$.
- Let $\text{Coll}^\Delta = \{F([0], c) \rightarrow F([0], t)\}$ for all c in \mathcal{C} .

The naming is done analogously to Θ -objects; the last set is meant to suggest collapse maps. Let $\mathcal{S}^\Delta = \text{Se}^\Delta \cup \text{Cpt}^\Delta \cup \text{Rec}^\Delta(\mathcal{S}) \cup \text{Coll}^\Delta$. Then X is a complete Segal object if and only if it is injective fibrant and \mathcal{S}^Δ -local.

Localizing the injective model structure with respect to \mathcal{S}^Δ produces the desired model structure. \square

We have the following result which allows us to identify complete Segal objects.

Proposition 5.7. *Let $f: W \rightarrow Z$ be a map between \mathcal{S}^Δ -local objects in $s\mathcal{P}Sh(\Delta \times \mathcal{C})$. Then f is in $\overline{\mathcal{S}^\Delta}$ if and only if*

$$f([0], t): W([0], t) \rightarrow Z([0], t)$$

and

$$f([1], c): W([1], c) \rightarrow Z([1], c)$$

for all c in \mathcal{C} are weak equivalences of spaces.

Proof. Recall that a map between \mathcal{S}^Δ -fibrant objects is in $\overline{\mathcal{S}^\Delta}$ if and only if it is a levelwise weak equivalence. For \mathcal{S}^Δ -fibrant objects, the value at a general $([m], c)$ in $\Delta \times \mathcal{C}$ is a homotopy limit of values at $([1], c)$ and $([0], t)$, so the proposition holds when W and Z are \mathcal{S}^Δ -fibrant. Since any \mathcal{S}^Δ -local object is levelwise weakly equivalent to a \mathcal{S}^Δ -fibrant object, the proposition follows. \square

We now turn to model structures on $s\mathcal{PSh}(\Delta \times \mathcal{C})$ where only some of the conditions for complete Segal objects hold.

Proposition 5.8. *There is a model structure on the category $s\mathcal{PSh}(\Delta \times \mathcal{C})$ in which the fibrant objects satisfy conditions (1)-(4).*

Proof. The existence of the model structure can be obtained by localizing the Segal object model structure with respect to Cpt^Δ . \square

However, this model structure as described is not cartesian. Therefore, we include the following result.

Proposition 5.9. *There is a cartesian model structure on the category $s\mathcal{PSh}(\Delta \times \Theta_n)$ in which the fibrant objects satisfy conditions (1)-(3) and (4').*

Proof. The model structure can be obtained by localizing the Segal object model structure with respect to the set of maps

$$\{F[0] \times Fc \rightarrow E \times Fc\}$$

where c ranges over all objects of \mathcal{C} . The proof that the model structure is cartesian can be proved similarly as the Segal object model structure was in Theorem 5.2. \square

6. COMPARISON BETWEEN COMPLETE SEGAL OBJECTS AND Θ -OBJECTS

In this section we establish a Quillen equivalence between the model structures for Θ -objects and the model structure for complete Segal objects.

Proposition 6.1. *The functor $d_*: s\mathcal{PSh}(\Delta \times \mathcal{C}) \rightarrow s\mathcal{PSh}(\Theta\mathcal{C})$ takes \mathcal{S}^Δ -fibrant objects to \mathcal{S}^Θ -fibrant objects.*

Proof. Given a complete Segal object W , we must check conditions (1),(2), (3), and (4) for d_*W to be a Θ -object. Since d_* is the right adjoint of a Quillen pair between injective model categories, d_*W is injective fibrant, establishing (1).

Recall that $\text{Map}_{s\mathcal{PSh}(\Theta\mathcal{C})}(X, d_*W) \cong \text{Map}_{s\mathcal{PSh}(\Delta \times \mathcal{C})}(d^*X, W)$. Thus, to show that (2), (3), and (4) hold, it suffices to show that $d^*\text{Se}^\Theta$, $d^*\text{Cpt}^\Theta$, and $d^*\text{Rec}^\Theta(\mathcal{S})$ are all contained in $\overline{\mathcal{S}^\Delta}$.

We note that to show that a map f is contained in $\overline{\mathcal{S}^\Delta}$, it suffices to show that f is weakly equivalent (with respect to levelwise weak equivalences in $s\mathcal{PSh}(\Delta \times \mathcal{C})$) to a homotopy colimit $\text{hocolim } f_\alpha$, where each $f_\alpha \in \overline{\mathcal{S}^\Delta}$, and the homotopy colimit is computed with respect to the levelwise weak equivalences.

We further note that if $B \xleftarrow{f} A \rightarrow C$ is a diagram in either $s\mathcal{PSh}(\Delta \times \mathcal{C})$ or $s\mathcal{PSh}(\Theta\mathcal{C})$ or $s\mathcal{PSh}(\Delta)$ such that f is an injective cofibration (i.e., a levelwise monomorphism), then the pushout of the diagram is also a homotopy pushout with respect to levelwise weak equivalences. Furthermore, d^* preserves such diagrams, being the left adjoint of a Quillen pair between injective model structures.

To prove (2), we want to show that $d^*\text{Se}^\Theta \subseteq \overline{\mathcal{S}^\Delta}$. Consider the maps

$$\varphi^{m,\mathcal{E}} = V_{\varphi^m}[m](Fc_1, \dots, Fc_m): V_{G[m]}[m](Fc_1, \dots, Fc_m) \rightarrow V[m](Fc_1, \dots, Fc_m)$$

in Se^Θ . We aim to show by induction on m that $d^*\varphi^{m,\underline{c}} \in \overline{\mathcal{S}^\Delta}$ for all $\underline{c} = c_1, \dots, c_m$. The cases of $m = 0$ and $m = 1$ are immediate, since φ^0 and $\varphi^{1,c}$ are isomorphisms.

Now suppose $m \geq 2$. We need to show that $d^*\varphi^{m,\underline{c}} \in \overline{\mathcal{S}^\Delta}$ by using the (homotopy) pushout square of Proposition 3.8. That is, we need to show that each of the three maps

$$\begin{aligned}\alpha &= (T_m \times_{F[m]} \varphi^m) \times (Fc_1 \times \dots \times Fc_m), \\ \beta &= \varphi^m \times (Fc_1 \times \dots \times Fc_m),\end{aligned}$$

and

$$\gamma = d^*(V_{T_m \times_{F[m]} s_m}[m](Fc_1, \dots, Fc_m)).$$

are contained in $\overline{\mathcal{S}^\Delta}$. (Recall that $T_m = F[0, m-1] \cup_{F[1, m-1]} F[1, m]$.)

Because Se^Δ is cartesian, we know that $\varphi^k \times Y \in \overline{\text{Se}^\Delta} \subseteq \overline{\mathcal{S}^\Delta}$ for all $k \geq 0$ and all Y in $s\mathcal{PSh}(\Delta \times \mathcal{C})$. Thus the map β is in $\overline{\mathcal{S}^\Delta}$.

It is straightforward to check that the pullback of the map $\varphi^m: G[m] \rightarrow F[m]$ along the inclusion $F[p, p+k] \rightarrow F[m]$ is isomorphic to the map $\varphi^k: G[k] \rightarrow F[k]$. We can use this observation to identify both α and γ as homotopy pushouts of maps in $\overline{\mathcal{S}^\Delta}$.

That is, the map α is itself a (homotopy) pushout of maps of the form $\varphi^k \times Y$, and thus is in $\overline{\mathcal{S}^\Delta}$. Similarly, the map γ is a (homotopy) pushout of maps of the form $V_{\varphi^k}[k](Fc_{i_1}, \dots, Fc_{i_k})$, which are contained in $\overline{\mathcal{S}^\Delta}$ by the inductive hypothesis, since $k < m$.

To prove (3), recall that $d^*\pi_\Theta^* = \pi_\Delta^*$, and thus $d^*\text{Cpt}^\Theta = \text{Cpt}^\Delta \subseteq \overline{\mathcal{S}^\Delta}$. Lastly, to prove (4), we have that $d^*V[1](f) = \Sigma(f)$, and thus $d^*\text{Rec}^\Theta(\mathcal{S}) \subseteq \text{Rec}^\Delta(\mathcal{S}) \subseteq \overline{\mathcal{S}^\Delta}$. \square

Corollary 6.2. *The adjoint pair*

$$d^*: s\mathcal{PSh}(\Theta\mathcal{C})_{\mathcal{S}^\Theta} \rightleftarrows s\mathcal{PSh}(\Delta \times \mathcal{C})_{\mathcal{S}^\Delta} : d_*$$

is a Quillen pair.

Proof. Using Proposition 6.1, we can conclude that $d^*(\overline{\mathcal{S}^\Theta}) \subseteq \overline{\mathcal{S}^\Delta}$, and thus d^* preserves both cofibrations and acyclic cofibrations. \square

Proposition 6.3. *The functor $d^*: s\mathcal{PSh}(\Theta\mathcal{C}) \rightarrow s\mathcal{PSh}(\Delta \times \mathcal{C})$ takes \mathcal{S}^Θ -local objects to \mathcal{S}^Δ -local objects.*

Proof. Given an \mathcal{S}^Θ -local object X of $s\mathcal{PSh}(\Theta\mathcal{C})$, we want to show that d^*X is \mathcal{S}^Δ -local. Since d^* preserves all levelwise weak equivalences, we can assume without loss of generality that X is also injective fibrant, i.e., that X is \mathcal{S}^Θ -fibrant. Then we want to show that d^*X is \mathcal{S}^Δ -local, or equivalently that an injective fibrant replacement $f: d^*X \rightarrow \mathcal{F}d^*X$ is \mathcal{S}^Δ -fibrant. Thus, we must show that $Y = \mathcal{F}_c d^*X$ satisfies conditions (2)–(5) for a complete Segal object.

First we prove (2). We have that

$$Y([m], c) \cong (d^*X)([m], c) = X([m](c, \dots, c)),$$

while

$$Y([1], c) \times_{Y([0], c)} \dots \times_{Y([0], c)} Y([1], c) \cong X[1](c) \times_{X[0]}^h \dots \times_{X[0]}^h X[1](c).$$

That (2) holds is then immediate from the fact that X is Se^Θ -fibrant.

Next we prove (3). We have that

$$Y([m], t) \cong (d^* X)([m], t) = X[m](t, \dots, t)$$

for all m . That is, there is a levelwise weak equivalence $\tau_\Theta^* X \rightarrow \tau_\Delta^* Y$ in $s\mathcal{PSh}(\Delta)$. Then (3) follows from the fact that X is Cpt^Θ -fibrant.

Next we establish (4). For each object c in C , the map $Y([1], c) \rightarrow Y([0], c)^{\times 2}$ (which is a fibration since Y is injective fibrant and $\partial F[1] \times Fc \rightarrow F[1] \times Fc$ is an injective cofibration in $s\mathcal{PSh}(\Delta \times C)$) is weakly equivalent to the map $X([1](c)) \rightarrow X([0])^{\times 2}$. The latter map is also a fibration since X is injective fibrant and $F[0] \amalg F[0] \rightarrow F[1](c)$ is an injective cofibration in $s\mathcal{PSh}(\Theta C)$, and the map $X[0] \rightarrow Y([0], t)$ is a weak equivalence. Thus we have a levelwise weak equivalence

$$M_X^\Theta(x_0, x_1) = M_{d_* X}^\Delta(x_0, x_1) \rightarrow M_Y^\Delta(f(x_0), f(x_1))$$

in $s\mathcal{PSh}(C)$ for all $x_0, x_1 \in X[0]$. Now (4) follows from the fact that X is $\text{Rec}^\Theta(\mathcal{S})$ -fibrant.

Lastly, we prove (5). We have that $d^* X([0], c) = X([0])$, and thus $d^* X([0], t) \rightarrow d^* X([0], c)$ is an isomorphism for all c . Therefore $Y([0], t) \rightarrow Y([0], c)$ is a weak equivalence for all c . \square

Theorem 6.4. *The adjoint pair*

$$d^*: s\mathcal{PSh}(\Theta C)_{\mathcal{S}^\Theta} \rightleftarrows s\mathcal{PSh}(\Delta \times C)_{\mathcal{S}^\Delta} : d_*$$

is a Quillen equivalence.

Proof. Recall that all objects are cofibrant in both model categories. We must prove that:

- (1) for any \mathcal{S}^Δ -fibrant object W in $s\mathcal{PSh}(\Delta \times C)$, the map $d^* d_* W \rightarrow W$ is in $\overline{\mathcal{S}^\Delta}$, and
- (2) for any object X in $s\mathcal{PSh}(\Theta C)$, the composite

$$X \rightarrow d_* d^* X \rightarrow d_* \mathcal{F}_{\mathcal{S}^\Delta} d^* X$$

is in $\overline{\mathcal{S}^\Theta}$, where $\mathcal{F}_{\mathcal{S}^\Delta}$ is a fibrant replacement functor in $s\mathcal{PSh}(\Delta \times C)_{\mathcal{S}^\Delta}$.

First we prove (1). By Propositions 6.1 and 6.3, we know that $d^* d_* W$ is \mathcal{S}^Δ -local. Since the map in question is between two local objects, it suffices by Proposition 5.7 to show that

$$(d^* d_* W)([0], t) \rightarrow W([0], t), \quad \text{and} \quad (d^* d_* W)([1], c) \rightarrow W([1], c)$$

are weak equivalences. The first map is actually an isomorphism, using the fact that

$$\tau_\Delta^* d^* d_* \cong \tau_\Theta^* d_* \cong (\pi_\Theta)_* d_* \cong (\pi_\Delta)_* \cong \tau_\Delta^*.$$

For the second map, note that

$$(d^* d_* W)([1], c) \cong \text{Map}_{s\mathcal{PSh}(\Delta \times C)}(\Sigma(Fc), W),$$

and that the map can be identified in the top arrow of the pullback square

$$\begin{array}{ccc} \text{Map}(\Sigma(Fc), W) & \longrightarrow & \text{Map}(F[1] \times Fc, W) \\ \downarrow & & \downarrow \\ \text{Map}(\partial F[1] \times Ft, W) & \longrightarrow & \text{Map}(\partial F[1] \times Fc, W) \end{array}$$

The square is in fact a homotopy pullback square of spaces (since W is injective fibrant and $\partial F[1] \times Fc \rightarrow F[1] \times Fc$ is an injective cofibration). The result follows from the fact that the bottom horizontal arrow is a weak equivalence, since W is \mathcal{S}^Δ -fibrant.

Next we prove (2). Choose an \mathcal{S}^Θ -fibrant replacement $X \rightarrow \mathcal{F}_{\mathcal{S}^\Theta} X$, and consider the commutative square

$$\begin{array}{ccc} X & \longrightarrow & d_* \mathcal{F}_{\mathcal{S}^\Delta} d^* X \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{S}^\Theta} X & \longrightarrow & d_* \mathcal{F}_{\mathcal{S}^\Delta} d^* \mathcal{F}_{\mathcal{S}^\Theta} X \end{array}$$

in which the vertical arrows are in $\overline{\mathcal{S}^\Theta}$. For the right-hand arrow, we use the fact that

$$d^* : s\mathcal{P}Sh(\Theta\mathcal{C})_{c, \mathcal{S}^\Theta} \rightleftarrows s\mathcal{P}Sh(\Delta \times \mathcal{C})_{c, \mathcal{S}^\Delta} : d_*$$

is a Quillen pair. Thus, to show that the top horizontal arrow is in $\overline{\mathcal{S}^\Theta}$, it suffices to show the bottom horizontal arrow is. Furthermore, since d^* takes \mathcal{S}^Θ -local objects to \mathcal{S}^Δ -local objects by Proposition 6.3, we can replace the $\mathcal{F}_{\mathcal{S}^\Delta}$ in the lower right-hand corner with an injective fibrant replacement \mathcal{F} . The resulting object $d_* \mathcal{F} d^* \mathcal{F}_{\mathcal{S}^\Theta} X$ is itself \mathcal{S}^Θ -fibrant by Proposition 6.1.

Therefore, by Proposition 4.5 it suffices to prove that if X is a \mathcal{S}^Θ -fibrant object of $s\mathcal{P}Sh(\Theta\mathcal{C})$, then the maps

$$X[0] \rightarrow (d_* \mathcal{F} d^* X)[0], \quad X[1](c) \rightarrow (d_* \mathcal{F} d^* X)[1](c)$$

are weak equivalences of spaces. The result follows using Lemma 6.5. \square

Lemma 6.5. *Let X be any object of $s\mathcal{P}Sh(\Theta\mathcal{C})$. Then the maps*

$$X[0] \rightarrow (d_* \mathcal{F} d^* X)[0], \quad X[1](c) \rightarrow (d_* \mathcal{F} d^* X)[1](c)$$

are weak equivalences of spaces.

Proof. In the first case, because $d^*(F[0]) \cong F[0] \times Ft$, the map is isomorphic to the weak equivalence

$$(d^* X)([0], t) \rightarrow (\mathcal{F} d^* X)([0], t).$$

In the second case, because $d^*(F[1](c)) \cong \Sigma(Fc)$, the map is isomorphic to

$$\text{Map}(\Sigma(Fc), d^* X) \rightarrow \text{Map}(\Sigma(Fc), \mathcal{F} d^* X).$$

Using the pushout decomposition

$$\Sigma(Fc) \cong (F[1] \times Fc) \cup_{\partial F[1] \times Fc} \partial F[1] \times Ft,$$

we see that we can obtain this map by taking pullbacks of the rows in the diagram

$$\begin{array}{ccccc} (d^* X)([1], c) & \longrightarrow & (d^* X)([0], c)^{\times 2} & \xleftarrow[\simeq]{g} & (d^* X)([0], t)^{\times 2} \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ (\mathcal{F} d^* X)([1], c) & \xrightarrow{f} & (\mathcal{F} d^* X)([0], c)^{\times 2} & \longleftarrow & (\mathcal{F} d^* X)([0], t)^{\times 2} \end{array}$$

The vertical maps are weak equivalences of spaces. The map marked f is a fibration, since $\partial F[1] \times Fc \rightarrow F[1] \times Fc$ is an injective cofibration and $\mathcal{F} d^* X$ is injective fibrant. The map marked g is an isomorphism (and thus a fibration), since $d_\#(F[0] \times$

$Fc \rightarrow F[0] \times Ft$) is an isomorphism. Therefore, taking limits along rows computes homotopy pullbacks, and the induced map between them is a weak equivalence. \square

7. EQUIVALENCE BETWEEN $\Theta_n Sp$ AND COMPLETE SEGAL SPACE OBJECTS IN $\Theta_{n-1} Sp$

In this section we look at consequences of the results of the previous section for models of (∞, n) -categories.

Let $\Theta_n Sp$ denote the model category $sPSh(\Theta_n)_{S\Theta}$, and let $CSS(\Theta_{n-1} Sp)$ denote the model category $sPSh(\Delta \times \Theta_{n-1})_{S\Delta}$. Then we have the following special case of Theorem 6.4.

Corollary 7.1. *The adjoint pair (d^*, d_*) induces a Quillen equivalence of localized model categories*

$$d_* : CSS(\Theta_{n-1} Sp) \rightleftarrows \Theta_n Sp : d^*.$$

However, this Quillen equivalence can be extended to a chain, by iterating the application of the adjoints (d_*, d^*) .

Observe that the functor d can be iterated to obtain a chain of functors connecting Δ^n and Θ_n :

$$\Delta^n \rightarrow \Delta^{n-1} \times \Delta \rightarrow \Delta^{n-2} \times \Theta_2 \rightarrow \cdots \rightarrow \Delta \times \Theta_{n-1} \rightarrow \Theta_n.$$

This chain induces a string of Quillen pairs

$$sPSh(\Delta^n) \rightleftarrows sPSh(\Delta^{n-1} \times \Delta) \rightleftarrows \cdots \rightleftarrows sPSh(\Theta_n)$$

on the level of injective model structures.

Then applying Theorem 6.4 to each of these adjoint pairs, we have the following result.

Corollary 7.2. *For any $1 \leq i \leq n$, there exists a model structure $CSS^i(\Theta_{n-i} Sp)$ on the category $sPSh(\Delta^i \times \Theta_{n-i})$ in which the fibrant objects W satisfy the following conditions:*

- (1) W is injective fibrant,
- (2) $W(-, x)$ is a Segal space for each object x in $\Delta^{i-1} \times \Theta_{n-i}$,
- (3) $W([m], -)$ is a complete Segal object in $CSS^{i-1}(\Theta_{n-i} Sp)$ for each object $[m]$ of Δ ,
- (4) $W(-, t)$ is a complete Segal space, where t is the terminal object of $\Delta^{i-1} \times \Theta_{n-i}$, and
- (5) $W([0], -)$ is essentially constant..

Observe that in the case where $i = 0$, if we take $CSS^0(\Theta_{n-1} Sp)$ to be $\Theta_{n-1} Sp$, then we obtain complete Segal objects in $\Theta_{n-1} Sp$, as described in Corollary 7.1. Furthermore, if $i = n$, this description of the fibrant objects exactly coincides with the Barwick-Lurie definition of n -fold complete Segal spaces [15].

Corollary 7.3. *There is a chain of Quillen equivalences*

$$CSS^n(sSets) \rightleftarrows CSS^{n-1}(\Theta_1 Sp) \rightleftarrows \cdots \rightleftarrows CSS(\Theta_{n-1} Sp) \rightleftarrows \Theta_n Sp.$$

8. NOTIONS OF EQUIVALENCE IN COMPLETE SEGAL OBJECTS

The goal for the remainder of the paper is to establish the final desired Quillen equivalence of model categories, namely the one between Segal category objects and complete Segal objects. Ultimately, we must restrict to complete Segal objects in $s\mathcal{PSh}(\Delta \times \Theta_n)$, since currently we only have a model structure for Segal category objects in this context; we refer to these objects as complete Segal objects in $\Theta_n Sp$ and refer to the model structure as $\mathcal{CSS}(\Theta_n Sp)$. Most of the constructions to this section that we make in this setting can be done in the more general context, however.

It is often convenient here to think of (complete) Segal objects here as functors $\Delta^{op} \rightarrow \Theta_n Sp$, although at time we continue to think of them as functors $\Delta^{op} \times \Theta_n^{op} \rightarrow sSets$ as previously. In the latter perspective, when considering at the terminal objects of Δ and Θ_n , we sometimes clarify by denoting them by $[0]_\Delta$ and $[0]_\Theta$, respectively.

8.1. Segal objects in $\Theta_n Sp$. Given a Segal object W , define its *objects* to be the elements of the set $\text{ob}(W) = W([0]_\Delta, [0]_\Theta)_0$. Given two elements $x, y \in \text{ob}(W)$, we can define the *mapping object* $M_W^\Delta(x, y)$ as the fiber of the map $(d_1, d_0): W_1 \rightarrow W_0 \times W_0$ over the point (x, y) . Since W is assumed to be injective fibrant, which is precisely Reedy fibrant in this case, the mapping space is homotopy invariant. Observe that this definition coincides with the one given in Section 3.9.

As we have seen above, we can obtain a simplicial space from a Θ_n -space via the functor τ_Θ^* . Applying the diagonal functor to $\tau_\Theta^* M_W^\Delta(x, y)$, we get mapping spaces for W . As a consequence, it is possible to define the *homotopy category* $\text{Ho}(W)$ of a Segal object W , where the objects are those of W and the morphisms are the components of the mapping spaces.

Definition 8.2. A map $f: W \rightarrow Z$ of Segal objects is a *Dwyer-Kan equivalence* if:

- for any objects x and y of W , the induced map $M_W^\Delta(x, y) \rightarrow M_Z^\Delta(fx, fy)$ is a weak equivalence in $\Theta_n Sp$, and
- the induced functor $\text{Ho}(W) \rightarrow \text{Ho}(Z)$ is an equivalence of categories.

In order to compare complete Segal space objects to Segal category objects, we need to understand the relationship between the weak equivalences in the two model structures. The purpose of this section is to establish that weak equivalences between fibrant objects in $\mathcal{CSS}(\Theta_n Sp)$ are precisely Dwyer-Kan equivalences.

For any $x \in \text{ob}(W) = W([0]_\Delta, [0]_\Theta)_0$, define its *identity map* id_x to be $s_0(x) \in M_W^\Delta(x, x)_0$. Two maps $f, g \in M_W^\Delta(x, y)_0$ are *homotopic* if they lie in the same component of the induced mapping space, denoted by $f \simeq g$.

Recall the more general mapping object $M_W^\Delta(x_0, \dots, x_k)$ from Section 3.9; it comes with induced acyclic fibrations

$$\varphi_k: M_W^\Delta(x_0, \dots, x_k) \rightarrow M_W^\Delta(x_0, x_1) \times \cdots \times M_W^\Delta(x_{k-1}, x_k).$$

We can then define composition in a Segal object W . Given $(f, g) \in M_W^\Delta(x, y) \times M_W^\Delta(y, z)$, a *composition* is a lift of φ_2 to some $k \in M_W^\Delta(x, y, z)$; a *result* of this composition is $d_1(k) \in M_W^\Delta(x, z)$. While k is not uniquely determined, since φ_2 is an acyclic fibration we can conclude that any two choices for k give results that are homotopic. Therefore, we write $g \circ f$ for the result of some composition of f and g .

The following proposition can be proved as for ordinary Segal spaces [18, 5.4].

Proposition 8.3. *Let W be a Segal object. Given $f \in M_W^\Delta(w, x)$, $g \in M_W^\Delta(x, y)$, and $h \in M_W^\Delta(y, z)$, we have $(h \circ g) \circ f \simeq h \circ (g \circ f)$ and $f \circ \text{id}_w \simeq \text{id}_x \circ f$.*

Definition 8.4. Let W be a Segal object. An element $g \in M_W^\Delta(x, y)$ is a *homotopy equivalence* if there exist $f, h \in M_W^\Delta(y, x)$ such that $g \circ f \simeq \text{id}_y$ and $h \circ g \simeq \text{id}_x$.

Observe in this case that $h \simeq h \circ g \circ f \simeq f$ and that, for any $x \in \text{ob}(W)$, $\text{id}_x \in M_W^\Delta(x, x)$ is a homotopy equivalence. Once again, the following lemma can be proved just as for complete Segal spaces.

Proposition 8.5. *If $g \in W([1], [0]_\Theta)_0$ and there exists a path $F_\Theta[1]([0]) \rightarrow W_1$ from g to a homotopy equivalence $g' \in W([1], [0]_\Theta)_0$, then g is a homotopy equivalence.*

Definition 8.6. Define the *space of homotopy equivalences* W_{heq} to consist of those components of W_1 made containing homotopy equivalences.

Now observe that the degeneracy map $s_0: W_0 \rightarrow W_1$ factors through W_{heq} . We would like a complete Segal object in $\Theta_n \text{Sp}$ to satisfy the condition that the map $s_0: W_0 \rightarrow W_{\text{heq}}$ is a weak equivalence in $\Theta_n \text{Sp}$. As in the case of ordinary complete Segal spaces, we can show that this condition is equivalent to condition (4).

8.7. Categorical equivalences.

Definition 8.8. Let $f, g: U \rightarrow V$ be maps between Segal space objects. A *categorical homotopy* between f and g is given by a map $H: U \times E \rightarrow V$ such that the diagram

$$\begin{array}{ccc} U & & \\ \text{id} \times i_0 \downarrow & \searrow f & \\ U \times E & \xrightarrow{H} & V \\ \text{id} \times i_1 \uparrow & \nearrow g & \\ U & & \end{array}$$

commutes. Equivalently, a categorical homotopy is given by a map H' or H'' as given in its respective commutative diagram

$$\begin{array}{ccc} & V & \\ f \nearrow & \uparrow V^{i_0} & \\ U & \xrightarrow{H'} V^E & \\ g \searrow & \downarrow V^{i_1} & \\ & V & \end{array} \quad \begin{array}{ccc} & F & \\ i_0 \downarrow & \searrow \{f\} & \\ & E & \xrightarrow{H''} V^U \\ i_1 \uparrow & \nearrow \{g\} & \\ & F[0] & \end{array} .$$

Proposition 8.9. *Suppose that U and W are Segal objects and W satisfies condition (4'). Then maps $f, g: U \rightarrow W$ are categorically homotopic if and only if there exists a homotopy $K: U \rightarrow W^{F[1]}$ which restricts to f and g on its endpoints.*

Proof. Consider the following diagram in the category of functors $\Delta^{\text{op}} \rightarrow \Theta_n \text{Sp}$ over $W \times W$:

$$\begin{array}{ccc} W & \longrightarrow & W^E \\ \downarrow & & \downarrow \\ W^{F[1]} & \longrightarrow & W \times W \end{array}$$

where the maps out of W are given by the inclusion of an object into either $F[1]$ or E , as appropriate. The former is a levelwise weak equivalence since it is the inclusion of constant paths in the path fibration. The latter is a levelwise weak equivalence because W satisfies condition (4'). We know further that the maps to $W \times W$ are both fibrations in the Reedy model structure.

If f and g are simplicially homotopic, then we have a map $K: U \rightarrow W^{F[1]}$ such that the composite with the map to $W \times W$ is precisely (f, g) . To obtain a categorical homotopy, then we need this composite map to factor through W^E . If the maps out of W were fibrations (in particular if W were a pullback), then we could factor K through W . However, they are not.

To remedy this situation, take the pullback

$$P = W^{F[1]} \times_{W \times W} W^E$$

and then factor the natural map $W \rightarrow P$ as a levelwise acyclic cofibration followed by a fibration $W \rightarrow Q \rightarrow P$. Now the maps $W^{F[1]} \leftarrow P \rightarrow W^E$ are fibrations, since P is a pullback along fibrations, and by composition so are the maps $W^{F[1]} \leftarrow Q \rightarrow W^E$. But we also know that these two maps are levelwise weak equivalences, by the 2-out-of-3 property. Therefore, a lift

$$\begin{array}{ccc} & & Q \\ & \nearrow & \downarrow \simeq \\ U & \longrightarrow & W^{F[1]} \end{array}$$

exists, which we can compose with $Q \rightarrow W^E$ to obtain the desired categorical homotopy.

Now observe that this argument is completely symmetric; if we began instead with a categorical homotopy $U \rightarrow W^E$, we can produce a simplicial homotopy $U \rightarrow W^{F[1]}$. \square

Definition 8.10. A map $g: U \rightarrow V$ of Segal space objects is a *categorical equivalence* if there exist maps $f, h: V \rightarrow U$ together with categorical homotopies $gf \simeq \text{id}_V$ and $fg \simeq \text{id}_U$.

Proposition 8.11. A map $g: U \rightarrow V$ between Segal objects which satisfy (4') is a categorical equivalence if and only if it is a levelwise weak equivalence.

Proof. We know from Proposition 8.9 that g is a categorical equivalence if and only if it is a simplicial homotopy equivalence. Since U and V Reedy fibrant and cofibrant, then simplicial homotopy equivalence coincides with Reedy weak equivalence. \square

We now show that categorical equivalences are compatible with the cartesian structure.

Proposition 8.12. Let U, V , and W be Segal objects. If $f, g: U \rightarrow V$ are categorically homotopic maps, then so are the induced maps $W^f, W^g: W^V \rightarrow W^U$.

Proof. Let $H: U \times E \rightarrow V$ be a categorical homotopy between f and g . Then

$$W^H: W^V \rightarrow W^{U \times E} \cong (W^U)^E$$

defines a categorical homotopy between W^f and W^g . \square

Corollary 8.13. *If $U \rightarrow V$ is a categorical equivalence between Segal objects, then so is $W^V \rightarrow W^U$.*

Proposition 8.14. *A categorical equivalence $f: U \rightarrow V$ between Segal objects is a weak equivalence in the model structure of Theorem 5.9.*

Proof. We want to show that $\text{Map}(f, W)$ is a weak equivalence of simplicial sets for any Segal object W satisfying (4'). Consider $W^f: W^V \rightarrow W^U$. Since W is a fibrant object in a cartesian model structure, so are both W^V and W^U . Therefore, W^f is a categorical equivalence by Corollary 8.13 and therefore a levelwise weak equivalence by Proposition 8.11. But then passing to the mapping space $\text{Map}(f, W)$ is still a weak equivalence, proving the proposition. \square

8.15. Dwyer-Kan equivalences. The following result is the analogue of [18, 7.6]. Observe that so far in this section we have not used condition (5) for complete Segal space objects. However, when we consider Dwyer-Kan equivalences, it is necessary to use this property.

Proposition 8.16. *A map $f: U \rightarrow V$ between complete Segal objects is a Dwyer-Kan equivalence if and only if it is a levelwise weak equivalence.*

Proof. A Reedy weak equivalence is always a Dwyer-Kan equivalence, so we need only prove the reverse implication. Suppose that U and V are complete Segal objects and that $f: U \rightarrow V$ is a Dwyer-Kan equivalence. Then for any $x, y \in U([0]_\Delta, [0]_\Theta)$, we have that $M_U^\Delta(x, y) \rightarrow M_V^\Delta(fx, fy)$ is a weak equivalence in $\Theta_n Sp$, and that $\text{Ho}(U) \rightarrow \text{Ho}(V)$ is an equivalence of categories.

Recall that $M_U^\Delta(x, y)$ can be written as a (homotopy) pullback

$$\begin{array}{ccc} M_U^\Delta(x, y) & \longrightarrow & U_1 \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & U_0 \times U_0 \end{array}$$

and that U_{heq} consists of components of U_1 . Therefore, we can describe the mapping object of equivalences $M_U^\Delta(x, y)_{\text{heq}}$ by restricting U_1 to U_{heq} and taking the pullback

$$\begin{array}{ccc} M^{\text{Delta}}_U(x, y)_{\text{heq}} & \longrightarrow & U_{\text{heq}} \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & U_0 \times U_0. \end{array}$$

We obtain that for any pair of objects (x, y) , $M_U^\Delta(x, y)_{\text{heq}} \rightarrow M_V^\Delta(fx, fy)_{\text{heq}}$ is a weak equivalence in $\Theta_n Sp$.

By precomposing with the degeneracy map s_0 , we obtain a commutative diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & U_0 \times U_0 \\ \downarrow & & \downarrow \\ V_0 & \longrightarrow & V_0 \times V_0 \end{array}$$

which is a homotopy pullback diagram since taking the horizontal fibers results in a weak equivalence $M_U^\Delta(x, y)_{\text{heq}} \rightarrow M_V^\Delta(fx, fy)_{\text{heq}}$. Because we have assumed that U and V are complete Segal objects, so that U_0 and V_0 can be regarded as spaces,

it follows that the map $U_0 \rightarrow V_0$ is a weak equivalence. Then $U_1 \rightarrow V_1$ must also be a weak equivalence, since

$$\begin{array}{ccc} U_1 & \longrightarrow & U_0 \times U_0 \\ \downarrow & & \downarrow \\ V_1 & \longrightarrow & V_0 \times V_0 \end{array}$$

is a pullback diagram with horizontal maps fibrations, since U and V are Reedy fibrant. Using the Segal condition, we have established that f is a Reedy weak equivalence. \square

However, we need the following stronger result.

Theorem 8.17. *A map $f: U \rightarrow V$ of Segal objects is a Dwyer-Kan equivalence if and only if it is a weak equivalence in the model category $\mathcal{CSS}(\Theta_n Sp)$.*

The heart of the proof is in the following method for completing a Segal object.

Lemma 8.18. *Given any Segal space object W , there exists a completion map $i_W: W \rightarrow \widehat{W}$ such that:*

- (1) *the completion \widehat{W} is a complete Segal object;*
- (2) *the map i_W is a weak equivalence in $\mathcal{CSS}(\Theta_n Sp)$; and*
- (3) *the map i_W is a Dwyer-Kan equivalence.*

Proof. Generalizing the discrete object E above, let $E(k)$ denote the nerve of the category with $(k+1)$ objects and a single isomorphism between any two of them, regarded as a discrete simplicial object in Θ_n^{op} .

Since W is a Segal object, if we fix an object of Θ_n^{op} , we have a Segal space in the ordinary sense to which we can apply completion. Therefore, we simply define the completion of W to be given by the levelwise completion of Segal spaces to complete Segal spaces as defined in [18, §14]. More precisely, define the completion

$$\begin{aligned} \widehat{W} &= \text{diag}_\Delta \left([m] \mapsto W^{E(m)} \right) \\ &= \text{hocolim}_\Delta \left([m] \mapsto W^{E(m)} \right). \end{aligned}$$

Now observe that W itself can be regarded as a homotopy colimit as above, but with $E(m)$ replaced by $F[0]$; i.e., the homotopy colimit of the constant diagram given by W itself. We know that the unique map $E(m) \rightarrow \Delta[0]$ induces a categorical equivalence $W \rightarrow W^E$ which, by Proposition 8.14 is a weak equivalence in the model structure of Proposition 5.9. The map $W \rightarrow \widehat{W}$ is given by the induced map on homotopy colimits and is hence a weak equivalence.

It remains to show that the map $W \rightarrow \widehat{W}$ is a Dwyer-Kan equivalence. By construction, it is essentially surjective, so we need only check the condition on mapping objects. Consider the 0-coskeleton $\text{cosk}_0(W_0)$ which has as k -simplices the object W_0^{k+1} . We can apply Lemma 8.19 to the map $W \rightarrow \text{cosk}_0(W_0)$ and the homotopy pullback diagrams

$$\begin{array}{ccc} W_q & \longrightarrow & W_p \\ \downarrow & & \downarrow \\ W_0^{q+1} & \longrightarrow & W_0^{p+1} \end{array}$$

ranging over all $[p] \rightarrow [q]$ in Δ to conclude that

$$\begin{array}{ccc} W_k & \longrightarrow & \widehat{W}_k \\ \downarrow & & \downarrow \\ W_0^{k+1} & \longrightarrow & \widehat{W}_0^{k+1} \end{array}$$

is a homotopy pullback square. The homotopy fibers of the vertical maps are precisely the mapping objects of W and \widehat{W} , respectively, which are hence weakly equivalent. The case where $k = 2$ gives the required condition for a Dwyer-Kan equivalence.

We now modify \widehat{W} so that it satisfies condition (5). Consider the Θ_n -space \widehat{W}_0 , and the simplicial object $\text{cosk}_0(\widehat{W}_0)$, equipped with a natural map $\widehat{W} \rightarrow \text{cosk}_0(\widehat{W}_0)$. Similarly, think of $\widehat{W}([0]_\Delta, [0]_\Theta)$ as a constant Θ_n -space and take its coskeleton $\text{cosk}_0(\widehat{W}([0]_\Delta, [0]_\Theta))$. Define \widehat{W}^d to be the pullback of the diagram

$$\begin{array}{ccc} \widehat{W}^d & \longrightarrow & \text{cosk}_0(\widehat{W}([0]_\Delta, [0]_\Theta)) \\ \downarrow & & \downarrow \\ \widehat{W} & \longrightarrow & \text{cosk}_0(\widehat{W}_0). \end{array}$$

Then \widehat{W}^d is still a Segal object, still satisfies condition (4), and now additionally satisfies condition (5); it also comes with a natural completion map $i_W: W \rightarrow \widehat{W}^d$ which is still a Dwyer-Kan equivalence. \square

Lemma 8.19. *Let $X \rightarrow Y$ be a map of simplicial objects in $\Theta_n Sp$ such that, for every map $[p] \rightarrow [q]$ in Δ , the induced diagram*

$$\begin{array}{ccc} X_q & \longrightarrow & X_p \\ \downarrow & & \downarrow \\ Y_q & \longrightarrow & Y_p \end{array}$$

is a homotopy pullback square. Then the diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & \text{hocolim}_\Delta X \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & \text{hocolim}_\Delta Y \end{array}$$

is also a homotopy pullback square.

We are now able to prove the main result of this section.

Proof of Theorem 8.17. Let $f: U \rightarrow V$ be a map of Segal objects. Using the completion functor from Lemma 8.18, we have that f is a weak equivalence in $\mathcal{CSS}(\Theta_n Sp)$ if and only if $\widehat{f}: \widehat{U} \rightarrow \widehat{V}$ is a weak equivalence. But \widehat{f} is a map of complete Segal space objects, which are the fibrant objects in a localized model structure, so \widehat{f} is a weak equivalence if and only if it is a levelwise weak equivalence. But then by Proposition 8.16, \widehat{f} is a levelwise weak equivalence if and only if \widehat{f} is a Dwyer-Kan equivalence. Finally, since Dwyer-Kan equivalences satisfy

the 2-out-of-3 property we also know that f is a Dwyer-Kan equivalence if and only if \widehat{f} is. Therefore, f is a weak equivalence if and only if it is a Dwyer-Kan equivalence. \square

9. EQUIVALENCE BETWEEN SEGAL CATEGORY AND COMPLETE SEGAL SPACE OBJECTS IN $\Theta_n Sp$

With the results of the last section, we are now able to give the Quillen equivalence between the model structures for Segal category objects and complete Segal objects. We begin by recalling the model structure for Segal category objects from [9].

Definition 9.1. A Segal category object in $\Theta_n Sp$ is a Segal object X in $\Theta_n Sp$ such that X_0 is discrete.

Theorem 9.2. [9, 6.9] *There is a cofibrantly generated model structure $SeCat(\Theta_n Sp)$ on the category of functors $X: \Delta^{op} \rightarrow \Theta_n Sp$ with X_0 discrete, with structure defined as follows.*

- *Weak equivalences are the maps $f: X \rightarrow Y$ such that the induced map $LX \rightarrow LY$ is a Dwyer-Kan equivalence of Segal space objects.*
- *Cofibrations are the monomorphisms.*
- *Fibrations are the maps with the right lifting property with respect to the maps which are both cofibrations and weak equivalences.*
- *Fibrant objects are precisely the Segal category objects.*

Recall that the underlying objects of the model category for complete Segal space objects in $\Theta_n Sp$ are functors $\Delta^{op} \rightarrow \Theta_n Sp$, whereas the underlying objects of the model category for Segal category objects are those functors $\Delta^{op} \rightarrow \Theta_n Sp$ which take the object $[0]$ of Δ^{op} to a discrete object in $\Theta_n Sp$, i.e., just a set. Therefore, we have an inclusion functor

$$I: SeCat(\Theta_n Sp) \rightarrow CSS(\Theta_n Sp).$$

We aim to describe a right adjoint functor to I and show that this adjoint pair defines a Quillen equivalence of model categories.

Let W be an object of $CSS(\Theta_n Sp)$, thought of as a functor $\Delta^{op} \rightarrow \Theta_n Sp$. Consider the functor $U = \text{cosk}_0(W_0)$. Thinking instead of W as a functor $\Delta^{op} \times \Theta_n^{op} \rightarrow Sets$, define $V = \text{cosk}_0(W([0]_\Delta, [0]_\Theta)_0)$.

Using the natural map $W \rightarrow U$ and the inclusion $V \rightarrow W$, define RW to be the pullback

$$\begin{array}{ccc} RW & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & U. \end{array}$$

Since W is assumed to be Reedy fibrant, the map $W \rightarrow U$ is a fibration and therefore RW is actually a homotopy pullback. The diagrams U and V are still Segal spaces (and in particular Reedy fibrant) but in general they are no longer complete.

Observe that RW has discrete Θ_n -space in degree 0. Therefore, we have defined a functor

$$R: CSS(\Theta_n Sp) \rightarrow SeCat(\Theta_n Sp).$$

Lemma 9.3. *If W is a complete Segal object, then RW is a Segal category object.*

Proof. By construction, we know that RW has the necessary discreteness property in degree 0 since V does. As we observed above, both U and V are both Segal objects, and so in particular V is a Segal category object. Using the definition of coskeleton, we get a diagram of Θ_n -spaces in degree 1

$$\begin{array}{ccc} (RW)_1 & \longrightarrow & W([0]_\Delta, [0]_\Theta)_0 \times W([0]_\Delta, [0]_\Theta)_0 \\ \downarrow & & \downarrow \\ W_1 & \longrightarrow & W_0 \times W_0. \end{array}$$

Looking at the corresponding pullback diagram in degree 2,

$$\begin{array}{ccc} (RW)_2 & \longrightarrow & (W_{0,[0],0})^3 \\ \downarrow & & \downarrow \\ W_2 \simeq W_1 \times_{W_0} W_1 & \longrightarrow & (W_0)^3 \end{array}$$

we can see that $(RW)_2$ must be weakly equivalent to $(RW)_1 \times_{(RW)_0} (RW)_1$. Proceeding similarly in higher degrees, we have that RW satisfies the Segal condition. \square

Proposition 9.4. *The functor R is right adjoint to the inclusion functor I .*

Proof. We need to prove that, for any object Y in $\mathcal{S}eCat(\Theta_n Sp)$ and any object W in $\mathcal{C}SS(\Theta_n Sp)$,

$$\mathrm{Hom}_{\mathcal{S}eCat(\Theta_n Sp)}(Y, RW) \cong \mathrm{Hom}_{\mathcal{C}SS(\Theta_n Sp)}(IY, W).$$

Since I is just the inclusion functor, note that $IY = Y$.

Suppose we have a map $Y \rightarrow W$. Since Y_0 is assumed to be discrete, $Y_0 = Y([0]_\Delta, [0]_\Theta)_0$, viewed as a constant functor $\Theta_n^{op} \rightarrow sSets$. Therefore, if $V = \mathrm{cosk}_0(W([0]_\Delta, [0]_\Theta)_0)$ as before, the map $Y \rightarrow W$ factors uniquely through V . The universal property of pullbacks then gives a unique map $Y \rightarrow RW$. Thus, we have described the functor $\varphi: \mathrm{Hom}(Y, W) \rightarrow \mathrm{Hom}(Y, RW)$. We claim that φ is an isomorphism.

The map φ is surjective, since any map $Y \rightarrow RW$ arises from the composite $Y \rightarrow RW \rightarrow W$. Furthermore, it is injective because of the uniqueness constraints in the definition of φ . \square

Proposition 9.5. *The adjoint pair (I, R) defines a Quillen pair.*

Proof. First, observe that I preserves cofibrations, since in each model category the cofibrations are precisely the monomorphisms.

It remains to show that I preserves acyclic cofibrations. First recall that a map in $\mathcal{S}eCat(\Theta_n Sp)$ is weak equivalence precisely if the induced map on fibrant replacements is a Dwyer-Kan equivalence. But these fibrant replacements are Segal category objects, and therefore Segal space objects. By Corollary 8.17, this map must be a weak equivalence in $\mathcal{C}SS(\Theta_n Sp)$. Therefore, the inclusion functor I preserves weak equivalences, in particular acyclic cofibrations. \square

Theorem 9.6. *The adjoint pair (I, R) is a Quillen equivalence.*

Proof. Using the argument in the previous proof, we can see that I reflects weak equivalences, and in particular those between cofibrant objects. Thus, it only remains to show that, given any fibrant object W in $\mathcal{CSS}(\Theta_n Sp)$, the map $IRW = RW \rightarrow W$ is a weak equivalence in $\mathcal{CSS}(\Theta_n Sp)$. Observe that this map is just the left-hand vertical map j in the pullback diagram

$$\begin{array}{ccc} RW & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & U. \end{array}$$

Since both W and RW are Segal space objects, it suffices to verify that j is a Dwyer-Kan equivalence. By the construction of RW , observe that $\text{ob}(W) = \text{ob}(RW)$, and j is the identity map on object sets. For any $x, y \in \text{ob}(W)$, consider $M_{RW}^\Delta(x, y) \rightarrow M_W^\Delta(x, y)$. Using the pullback describing $(RW)_1$ and restricting to this mapping object, we see that this map must be a weak equivalence in $\Theta_n Sp$. \square

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